Inverse Problem for a Curved Quantum Guide

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Abstract

In this paper, we consider the Dirichlet Laplacian operator $-\Delta$ on a curved quantum guide in $\mathbb{R}^n$ $(n = 2, 3)$ with an asymptotically straight reference curve. We give uniqueness results for the inverse problem associated to the reconstruction of the curvature by using either observations of spectral data or a boot-strapping method.
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1 Introduction and main results in dimension $n = 2$

The spectral properties of curved quantum guides have been studied intensively for several years, because of their applications in quantum mechanics, electron motion. We can cite among several papers [6], [8], [9], [4], [5], [3] ...
However, inverse problems associated with curved quantum guides have not been studied to our knowledge, except in [2]. Our aim is to establish uniqueness results for the inverse problem of the reconstruction of the curvature of the quantum guide: the data of one eigenpair determines uniquely the curvature up to its sign and similar results are obtained by considering the knowledge of a solution of Poisson's equation in the guide.
We consider the Laplacian operator on a non trivially curved quantum guide $\Omega \subset \mathbb{R}^2$ which is not self-intersecting, with Dirichlet boundary conditions, denoted by $-\Delta_D^\Omega$. We proceed as in [6]. We denote by $\Gamma = (\Gamma_1, \Gamma_2)$ the function C^3-smooth (see [3, Remark 5]) which characterizes the reference curve and by $N = (N_1, N_2)$ the outgoing normal to the boundary of Ω. We denote by d the fixed width of Ω and by $\Omega_0 := \mathbb{R}\times] - d/2, d/2[$. Each point (x, y) of Ω is described by the curvilinear coordinates (s, u) as follows:

$$\hat{f} : \Omega_0 \longrightarrow \Omega \quad \text{with} \quad (x, y) = \hat{f}(s, u) = \Gamma(s) + uN(s). \tag{1.1}$$

We assume $\Gamma_1'(s)^2 + \Gamma_2'(s)^2 = 1$ and we recall that the signed curvature γ of Γ is defined by:

$$\gamma(s) = -\Gamma_1''(s)\Gamma_2'(s) + \Gamma_2''(s)\Gamma_1'(s), \tag{1.2}$$

named so because $|\gamma(s)|$ represents the curvature of the reference curve at s. We recall that a guide is called simply-bent if γ does not change sign in $\mathbb{R}$. We assume throughout this article that:

Assumption 1.1. *i) $\hat{f}$ is injective.*

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ii) $\gamma \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\gamma \not\equiv 0$, *(i.e.* Ω *is non-trivially curved).*

iii) $\frac{d}{2} < \frac{1}{\|\gamma\|_\infty}$, *where* $\|\gamma\|_\infty := \sup_{s\in\mathbb{R}} |\gamma(s)| = \|\gamma\|_{L^\infty(\mathbb{R})}$.

iv) $\gamma(s) \to 0$ *as* $|s| \to +\infty$ *(i.e.* Ω *is asymptotically straight).*

Note that, by the inverse function theorem, the map $\hat{f}$ (defined by (1.1)) is a local diffeomorphism provided $1 - u\gamma(s) \neq 0$, for all u, s, which is guaranteed by Assumption 1.1 and since $\hat{f}$ is assumed to be injective, the map $\hat{f}$ is a global diffeomorphism. Note also that $1-u\gamma(s) > 0$ for all u and s. (More precisely, $0 < 1 - \frac{d}{2}\|\gamma\|_\infty \leq 1 - u\gamma(s) \leq 1 + \frac{d}{2}\|\gamma\|_\infty$ for all u, s.) The curvilinear coordinates (s, u) are locally orthogonal, so by virtue of the Frenet-Serret formulae, the metric in Ω is expressed with respect to them through a diagonal metric tensor (e.g. [9])

$$(g_{ij}) = \begin{pmatrix} (1 - u\gamma(s))^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

The transition to the curvilinear coordinates represents an isometric map of $L^2(\Omega)$ to $L^2(\Omega_0, g^{1/2}\ dsdu)$ where

$$(g(s, u))^{1/2} := 1 - u\gamma(s) \tag{1.4}$$

is the Jacobian $\dfrac{\partial(x, y)}{\partial(s, u)}$. So we can replace the Laplacian operator $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ by the Laplace-Beltrami operator H_g acting on $L^2(\Omega_0, g^{1/2} dsdu)$ relative to the given metric tensor (g_{ij}) (see (1.3) and (1.4)) where:

$$H_g := -g^{-1/2}\partial_s(g^{-1/2}\partial_s) - g^{-1/2}\partial_u(g^{1/2}\partial_u). \tag{1.5}$$

We rewrite H_g (defined by (1.5)) into a Schrödinger-type operator acting on $L^2(\Omega_0, dsdu)$. Indeed, using the unitary transformation

$$\begin{array}{rcl} U_g : L^2(\Omega_0, g^{1/2}\ dsdu) & \longrightarrow & L^2(\Omega_0, dsdu) \\ \psi & \mapsto & g^{1/4}\psi \end{array} \tag{1.6}$$

setting

$$H_\gamma := U_g H_g U_g^{-1},$$

we get

$$H_\gamma = -\partial_s(c_\gamma(s, u)\partial_s) - \partial_u^2 + V_\gamma(s, u) \tag{1.7}$$

with

$$c_\gamma(s, u) = \frac{1}{(1 - u\gamma(s))^2} \tag{1.8}$$

and

$$V_\gamma(s, u) = -\frac{\gamma^2(s)}{4(1 - u\gamma(s))^2} - \frac{u\gamma''(s)}{2(1 - u\gamma(s))^3} - \frac{5u^2\gamma'^2(s)}{4(1 - u\gamma(s))^4}. \tag{1.9}$$

We will assume throughout all this paper that the following assumption is satisfied:

Assumption 1.2. $\gamma \in C^2(\mathbb{R})$ *and* $\gamma^{(k)} \in L^\infty(\mathbb{R})$ *for each* $k = 0, 1, 2$ *where* $\gamma^{(k)}$ *denotes the* k^{th} *derivative of* γ.

Remarks: Since Ω is non trivially-curved and asymptotically straight, the operator $-\Delta_D^\Omega$ has at least one eigenvalue of finite multiplicity below its essential spectrum (see [3], [9] ; see also [6] under the additional assumptions that the width d is sufficiently small and the curvature γ is rapidly decaying at infinity ; see [8] under the assumption that the curvature γ has a compact

support).
Furthermore, note that such operator H_γ admits bound states and that the minimum eigenvalue λ_1 is simple and associated with a positive eigenfunction ϕ_1 (see [7, Sec.8.17]). Then, note that by [11, Theorem 7.1] any eigenfunction of H_γ is continuous and by [1, Remark 25 p.182] any eigenfunction of H_γ belongs to $H^2(\Omega_0)$.
Finally, note also that (λ, ϕ) is an eigenpair (i.e. an eigenfunction associated with its eigenvalue) of the operator H_γ acting on $L^2(\Omega_0, dsdu)$ means that $(\lambda, U_g^{-1}\phi)$ is an eigenpair of $-\Delta_D^\Omega$ acting on $L^2(\Omega)$. So the data of one eigenfunction of the operator H_γ is equivalent to the data of one eigenfunction of $-\Delta_D^\Omega$.

We first prove that the data of one eigenpair determines uniquely the curvature.

Theorem 1.1. *Let Ω be the curved guide in $\mathbb{R}^2$ defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1, 1.2. Let H_γ be the operator defined by (1.7) and (λ, ϕ) be an eigenpair of H_γ.*
Then

$$\gamma^2(s) = -4\frac{\Delta\phi(s,0)}{\phi(s,0)} - 4\lambda$$

for all s when $\phi(s,0) \neq 0$.

Note that the condition $\phi(s,0) \neq 0$ in Theorem 1.1 is satisfied for the positive eigenfunction ϕ_1 and for all $s \in \mathbb{R}$. Then, we prove under

Assumption 1.3. $\gamma \in C^5(\mathbb{R})$ *and* $\gamma^{(k)} \in L^\infty(\mathbb{R})$ *for each* $k = 0, \dots, 5$,

that one weak solution ϕ of the problem

$$\left\{ \begin{array}{l} H_\gamma \phi = f \text{ in } \Omega_0 \\ \phi = 0 \text{ on } \partial\Omega_0 \end{array} \right. \tag{1.10}$$

(where f is a known given function) is in fact a classical solution and the data of ϕ determines uniquely the curvature γ.

Theorem 1.2. *Let Ω be the curved guide in $\mathbb{R}^2$ defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1 and 1.3. Let H_γ be the operator defined by (1.7). Let $f \in H^3(\Omega_0) \cap C(\Omega_0)$ and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of (1.10).*
Then we have $\gamma^2(s) = -4\dfrac{\Delta\phi(s,0)}{\phi(s,0)} - 4\dfrac{f(s,0)}{\phi(s,0)}$ *for all s when $\phi(s,0) \neq 0$*

In the case of a simply-bent guide (i.e. when γ does not change sign in $\mathbb{R}$), we can restrain the hypotheses upon the regularity of γ. We obtain the following result:

Theorem 1.3. *Let Ω be the curved guide in $\mathbb{R}^2$ defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1 and 1.2. We assume also that γ is a nonnegative function. Let H_γ be the operator defined by (1.7). Let $f \in L^2(\Omega_0)$ be a non null function and let ϕ be a weak solution in $H_0^1(\Omega_0)$ of (1.10) Assume that there exists a positive constant M such that $|f(s,u)| \leq M|\phi(s,u)|$ almost everywhere in Ω_0. Then (f, ϕ) determines uniquely the curvature γ.*

Note that the above result is still valid for a nonpositive function γ.

This paper is organized as follows: In Section 2, we prove Theorems 1.1, 1.2 and 1.3. In Sections 3 and 4, we extend our results to the case of a curved quantum guide defined in $\mathbb{R}^3$.

2 Proofs of Theorems 1.1, 1.2 and 1.3

2.1 Proof of Theorem 1.1

Recall that ϕ is an eigenfunction of H_γ, belonging to $H^2(\Omega_0)$. Since ϕ is continuous and $H_\gamma\phi = \lambda\phi$, then $H_\gamma\phi$ is continuous too. Thus, noticing that $c_\gamma(s,0) = 1$, we deduce the continuity of the function $(s,0) \mapsto \Delta\phi(s,0)$ and from (1.7) to (1.9), we get:

$$-\Delta\phi(s,0) - \frac{\gamma^2(s)}{4}\phi(s,0) = \lambda\phi(s,0)$$

and equivalently,

$$\gamma^2(s) = -4\frac{\Delta\phi(s,0)}{\phi(s,0)} - 4\lambda \text{ if } \phi(s,0) \neq 0.$$

2.2 Proof of Theorem 1.2

First, we recall from [1, Remark 25 p.182] the following lemma.

Lemma 2.1. *For a second-order elliptic operator defined in a domain $\omega \subset \mathbb{R}^n$, if $\phi \in H_0^1(\omega)$ satisfies*

$$\int_\omega \sum_{i,j} a_{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j} = \int_\omega f\psi \textit{ for all } \psi \in H_0^1(\omega)$$

then if ω is of class C^2

$$(f \in L^2(\omega),\, a_{ij} \in C^1(\overline{\omega}),\, D^\alpha a_{ij} \in L^\infty(\omega) \textit{ for all } i,j \textit{ and for all } \alpha, |\alpha| \leq 1)$$

$$\textit{imply } (\phi \in H^2(\omega))$$

and for $m \geq 1$, if ω is of class C^{m+2}

$$(f \in H^m(\omega),\, a_{ij} \in C^{m+1}(\overline{\omega}),\, D^\alpha a_{ij} \in L^\infty(\omega) \textit{ for all } i,j \textit{ and for all } \alpha, |\alpha| \leq m+1)$$

$$\textit{imply } (\phi \in H^{m+2}(\omega)).$$

Now we can prove the Theorem 1.2.

We have $H_\gamma\phi = f$, so

$$\int_{\Omega_0} [c_\gamma(\partial_s\phi)(\partial_s\psi) + (\partial_u\phi)(\partial_u\psi)] = \int_{\Omega_0} [f - V_\gamma\phi]\psi \text{ for all } \psi \in H_0^1(\Omega_0) \tag{2.1}$$

with c_γ defined by (1.8) and V_γ defined by (1.9).

Using Assumption 1.3, since $\gamma^{(k)} \in L^\infty(\Omega_0)$ for $k = 0,1,2$ then $V_\gamma \in L^\infty(\Omega_0)$ and $f - V_\gamma\phi \in L^2(\Omega_0)$. From the hypotheses $\gamma \in C^1(\mathbb{R})$ and $\gamma' \in L^\infty(\mathbb{R})$, we get that $c_\gamma \in C^1(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for any α, $|\alpha| \leq 1$, and so, using Lemma 2.1 for the equation (2.1), we obtain that $\phi \in H^2(\Omega_0)$.
By the same way, we get that $f - V_\gamma\phi \in H^1(\Omega_0)$, $c_\gamma \in C^2(\overline{\Omega_0})$ and $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for any α, $|\alpha| \leq 2$ (from $\gamma \in C^3(\mathbb{R})$, $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for any $k = 0,\ldots,3$). Using Lemma 2.1, we obtain that $\phi \in H^3(\Omega_0)$.
We apply again the Lemma 2.1 to get that $\phi \in H^4(\Omega_0)$ (since $f - V_\gamma\phi \in H^2(\Omega_0)$, $c_\gamma \in C^3(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for all α, $|\alpha| \leq 3$, from the hypotheses $\gamma \in C^4(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$

for $k = 0, \dots, 4$.).
Finally, using Assumption 1.3 and Lemma 2.1, we obtain that $\phi \in H^5(\Omega_0)$.

Due to the regularity of Ω_0, we have $\phi \in H^5(\mathbb{R}^2)$ and $\Delta\phi \in H^3(\mathbb{R}^2)$. Since $\nabla(\Delta\phi) \in (H^2(\mathbb{R}^2))^2$ and $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, we can deduce that $\Delta\phi$ is continuous (see [1, Remark 8 p.154]).
Therefore we can conclude by using the continuity of the function

$$(s,0) \mapsto -\partial_s(c_\gamma(s,0)\partial_s\phi(s,0)) - \partial_u^2\phi(s,0) = f(s,0) - V_\gamma(s,0)\ \phi(s,0).$$

Therefore, we get: $-\Delta\phi(s,0) - \frac{\gamma^2(s)}{4}\phi(s,0) = f(s,0)$ and equivalently,

$$\gamma^2(s) = -4\frac{\Delta\phi(s,0)}{\phi(s,0)} - 4\frac{f(s,0)}{\phi(s,0)} \text{ if } \phi(s,0) \neq 0.$$

2.3 Proof of Theorem 1.3

We prove here that (f,ϕ) determines uniquely γ when γ is a nonnegative function.
For that, assume that Ω_1 and Ω_2 are two quantum guides in $\mathbb{R}^2$ with same width d. We denote by γ_1 and γ_2 the curvatures respectively associated with Ω_1 and Ω_2 and we suppose that each γ_i satisfies Assumption 1.2 and is a nonnegative function. Assume that $H_{\gamma_1}\phi = f = H_{\gamma_2}\phi$.
Then ϕ satisfies

$$-\partial_s((c_{\gamma_1}(s,u) - c_{\gamma_2}(s,u))\partial_s\phi(s,u)) + (V_{\gamma_1}(s,u) - V_{\gamma_2}(s,u))\phi(s,u) = 0. \tag{2.2}$$

Assume that $\gamma_1 \not\equiv \gamma_2$.
Step 1. First, we consider the case where (for example) $\gamma_1(s) < \gamma_2(s)$ for all $s \in \mathbb{R}$.
Let $\epsilon > 0$, $\omega_\epsilon := \mathbb{R} \times I_\epsilon$ with $I_\epsilon =]-\epsilon, 0[$. Multiplying (2.2) by ϕ and integrating over ω_ϵ, we get:

$$\int_{\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)^2 - \int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)\phi\nu_s + \int_{\omega_\epsilon} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0. \tag{2.3}$$

Since $\epsilon << 1$, $V_{\gamma_i}(s,u) \simeq -\frac{\gamma_i^2(s)}{4}$ for $i = 1, 2$, and so $V_{\gamma_1}(s,u) - V_{\gamma_2}(s,u) > 0$ in ω_ϵ.
Moreover, since

$$c_{\gamma_1}(s,u) - c_{\gamma_2}(s,u) = \frac{u(\gamma_1(s) - \gamma_2(s))(2 - u(\gamma_1(s) + \gamma_2(s))}{(1 - u\gamma_1(s))^2(1 - u\gamma_2(s))^2}, \tag{2.4}$$

we have $c_{\gamma_1}(s,u) > c_{\gamma_2}(s,u)$ in ω_ϵ.
Since

$$\int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)\phi\nu_s = 0, \tag{2.5}$$

Thus from (2.3)-(2.5), we get

$$\int_{\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)^2 + \int_{\omega_\epsilon} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0 \tag{2.6}$$

with $c_{\gamma_1} - c_{\gamma_2} > 0$ in ω_ϵ and $V_{\gamma_1} - V_{\gamma_2} > 0$ in ω_ϵ. We can deduce that $\phi = 0$ in ω_ϵ.
Using a unique continuation theorem (see [10, Theorem XIII.63 p.240]), from $H_\gamma\phi = f$, noting that $-\Delta(U_g^{-1}\phi) = U_g^{-1}f = g^{-1/4}f$, (recall that U_g is defined by (1.6)) and so by $|f| \leq M|\phi|$ we have $|\Delta(U_g^{-1}\phi)| \leq M|g^{-1/4}\phi|$ with $g > 0$ a.e., and we can deduce that $\phi = 0$ in Ω_0. So we get a contradiction (since $H_\gamma\phi = f$ and f is assumed to be a non null function).

Step 2. From Step 1, we obtain that there exists at least one point $s_0 \in \mathbb{R}$ such that $\gamma_1(s_0) = \gamma_2(s_0)$. Since $\gamma_1 \not\equiv \gamma_2$, we can choose $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that (for example) $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(s) < \gamma_2(s)$ for all $s \in]a, b[$ and $\gamma_1(b) = \gamma_2(b)$ if $b \in \mathbb{R}$.
We proceed as in Step 1, considering, in this case, $\omega_\epsilon :=]a, b[\times I_\epsilon$. We study again the equation (2.3) and as in Step 1, we have

$$\int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s \phi)\phi\nu_s = 0.$$

Indeed from (2.4) and $\gamma_1(a) = \gamma_2(a)$ we have $c_{\gamma_1}(a, u) = c_{\gamma_2}(a, u)$ and so

$$\int_{-\epsilon}^{0} (c_{\gamma_1}(a, u) - c_{\gamma_2}(a, u))\partial_s\phi(a, u)\phi(a, u)\ du = 0.$$

By the same way if $b \in \mathbb{R}$, we also have $c_{\gamma_1}(b, u) = c_{\gamma_2}(b, u)$. Thus the equation (2.3) becomes (2.6) with $c_{\gamma_1} - c_{\gamma_2} > 0$ in ω_ϵ and $V_{\gamma_1} - V_{\gamma_2} > 0$ in ω_ϵ. So $\phi = 0$ in ω_ϵ and as in Step 1, by a unique continuation theorem, we obtain that $\phi = 0$ in Ω_0. Therefore we get a contradiction.

Note that the previous theorem is true if we replace the hypothesis "γ is nonnegative" by the hypothesis "γ is nonpositive". Indeed, in this last case, we just have to take $I_\epsilon =]0, \epsilon[$ and the proof rests valid.

3 Uniqueness result for a $\mathbb{R}^3$-quantum guide

Now, we apply the same ideas for a tube Ω in $\mathbb{R}^3$. We proceed here as in [3]. Let $s \mapsto \Gamma(s)$, $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$, be a curve in $\mathbb{R}^3$. We assume that $\Gamma : \mathbb{R} \to \mathbb{R}^3$ is a C^4-smooth curve satisfying the following hypotheses

Assumption 3.1. Γ *possesses a positively oriented Frenet frame* $\{e_1, e_2, e_3\}$ *with the properties that*

i) $e_1 = \Gamma'$,

ii) $\forall i \in \{1, 2, 3\},\ e_i \in C^1(\mathbb{R}, \mathbb{R}^3)$,

iii) $\forall i \in \{1, 2\},\ \forall s \in \mathbb{R},\ e_i'(s)$ *lies in the span of* $e_1(s), \ldots, e_{i+1}(s)$.

Recall that a sufficient condition to ensure the existence of the Frenet frame of Assumption 3.1 is to require that for all $s \in \mathbb{R}$ the vectors $\Gamma'(s), \Gamma''(s)$ are linearly independent.
Then we define the moving frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ along Γ by following [3]. This moving frame better reflects the geometry of the curve and it is still called the Tang frame because it is a generalization of the Tang frame known from the theory of three-dimensional waveguides.
Given a C^5 bounded open connected neighborhood ω of $(0, 0) \in \mathbb{R}^2$, let Ω_0 denote the straight tube $\mathbb{R} \times \omega$. We define the curved tube Ω of cross-section ω about Γ by

$$\Omega := \tilde{f}(\mathbb{R} \times \omega) = \tilde{f}(\Omega_0),\ \tilde{f}(s, u_2, u_3) := \Gamma(s) + \sum_{i=2}^{3} u_i \sum_{j=2}^{3} R_{ij}(s)e_j(s) = \Gamma(s) + \sum_{i=2}^{3} u_i\tilde{e}_i(s) \quad (3.1)$$

with $u = (u_2, u_3) \in \omega$ and

$$R(s) := (R_{ij}(s))_{i,j\in\{2,3\}} = \begin{pmatrix} \cos(\theta(s)) & -\sin(\theta(s)) \\ \sin(\theta(s)) & \cos(\theta(s)) \end{pmatrix},$$

θ being a real-valued differentiable function such that $\theta'(s) = \tau(s)$ the torsion of Γ. This differential equation is a consequence of the definition of the moving Tang frame (see [3, Remark 3]).

Note that R is a rotation matrix in $\mathbb{R}^2$ chosen in such a way that (s, u_2, u_3) are orthogonal "coordinates" in Ω. Let k be the first curvature function of Ω. Recall that since $\Omega \subset \mathbb{R}^3$, k is a nonnegative function. We assume throughout all this section that the following hypothesis holds:

Assumption 3.2. *i)* $k \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $a := \sup_{u\in\omega} \|u\|_{\mathbb{R}^2} < \frac{1}{\|k\|_\infty}$, $k(s) \to 0$ *as* $|s| \to +\infty$

ii) Ω *does not overlap.*

The Assumption 3.2 assures that the map $\tilde{f}$ (defined by (3.1)) is a diffeomorphism (see [3]) in order to identify Ω with the Riemannian manifold $(\Omega_0, (g_{ij}))$ where (g_{ij}) is the metric tensor induced by $\tilde{f}$, i.e. $(g_{ij}) :=^t J(\tilde{f}).J(\tilde{f})$, ($J(\tilde{f})$ denoting the Jacobian matrix of $\tilde{f}$). Recall that $(g_{ij}) = diag(h^2, 1, 1)$ (see [3]) with

$$h(s, u_2, u_3) := 1 - k(s)(\cos(\theta(s))u_2 + \sin(\theta(s))u_3). \tag{3.2}$$

Note that Assumption 3.2 implies that $0 < 1 - a\|k\|_\infty \leq 1 - h(s, u_2, u_3) \leq 1 + a\|k\|_\infty$ for all $s \in \mathbb{R}$ and $u = (u_2, u_3) \in \omega$. Moreover, setting

$$g := h^2 \tag{3.3}$$

we can replace the Dirichlet Laplacian operator $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ by the Laplace-Beltrami operator K_g acting on $L^2(\Omega_0, hdsdu)$ relative to the metric tensor (g_{ij}). We can rewrite K_g into a Schrödinger-type operator acting on $L^2(\Omega_0, dsdu)$. Indeed, using the unitary transformation

$$\begin{array}{rcl} W_g : L^2(\Omega_0, hdsdu) & \longrightarrow & L^2(\Omega_0, dsdu) \\ \psi & \mapsto & g^{1/4}\psi \end{array} \tag{3.4}$$

setting

$$H_k := W_g K_g W_g^{-1}, \tag{3.5}$$

we get

$$H_k = -\partial_s(h^{-2}\partial_s) - \partial_{u_2}^2 - \partial_{u_3}^2 + V_k \tag{3.6}$$

where ∂_s denotes the derivative relative to s and ∂_{u_i} denotes the derivative relative to u_i and with

$$V_k := -\frac{k^2}{4h^2} + \frac{\partial_s^2 h}{2h^3} - \frac{5(\partial_s h)^2}{4h^4}. \tag{3.7}$$

We assume also throughout all this section that the following hypotheses hold:

Assumption 3.3. *i)* $k' \in L^\infty(\mathbb{R})$, $k'' \in L^\infty(\mathbb{R})$

ii) $\theta \in C^2(\mathbb{R})$, $\theta' = \tau \in L^\infty(\mathbb{R})$, $\theta'' \in L^\infty(\mathbb{R})$.

Remarks: Note that, as for the 2-dimensional case, such operator H_k (defined by (3.2)-(3.7)) admits bound states and that the minimum eigenvalue λ_1 is simple and associated with a positive eigenfunction ϕ_1 (see [3, 7]). Still note that (λ, ϕ) is an eigenpair of the operator H_k acting on $L^2(\Omega_0, dsdu)$ means that $(\lambda, W_g^{-1}\phi)$ is an eigenpair of $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ (with W_g defined by (3.4)). Finally, note that by [11, Theorem 7.1] any eigenfunction of H_k is continuous and by [1, Remark 25 p.182] any eigenfunction of H_k belongs to $H^2(\Omega_0)$.

As for the 2-dimensional case, first we prove that the data of one eigenpair determines uniquely the curvature.

Theorem 3.1. *Let Ω be the curved guide in $\mathbb{R}^3$ defined as above. Let k be the first curvature function of Ω. Assume that Assumptions 3.1 to 3.3 are satisfied. Let H_k be the operator defined by (3.2)-(3.7) and (λ, ϕ) be an eigenpair of H_k.*
Then $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\lambda$ for all s when $\phi(s,0,0) \neq 0$.

Then, under

Assumption 3.4. *i) $k \in C^5(\mathbb{R}), k^{(i)} \in L^\infty(\mathbb{R})$ for all $i = 0, \dots, 5$*
ii) $\theta \in C^5(\mathbb{R}), \theta^{(i)} \in L^\infty(\mathbb{R})$ for all $i = 1, \dots, 5$

where $k^{(i)}$ (resp. $\theta^{(i)}$) denotes the i-th derivative of k (resp. of θ), we obtain the following result:

Theorem 3.2. *Let Ω be the curved guide in $\mathbb{R}^3$ defined as above. Let k be the first curvature function of Ω. Assume that Assumptions 3.1 to 3.4 are satisfied. Let H_k be the operator defined by (3.2)-(3.7). Let $f \in H^3(\Omega_0) \cap C(\Omega_0)$ and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of $H_k\phi = f$ in Ω_0.*
Then ϕ is a classical solution and $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\frac{f(s,0,0)}{\phi(s,0,0)}$ for all s when $\phi(s,0,0) \neq 0$.

Remarks: Recall that in $\mathbb{R}^3$, k is a nonnegative function and that the condition imposed on ϕ ($\phi(s,0,0) \neq 0$) in Theorems 3.1 and 3.2 is satisfied by the positive eigenfunction ϕ_1.

As for the two-dimensional case, we can restrain the hypotheses upon the regularity of the functions k and θ.

For a guide with a known torsion, we obtain the following result:

Theorem 3.3. *Let Ω be the curved guide in $\mathbb{R}^3$ defined as above. Let k be the first curvature function of Ω and let τ be the second curvature function (i.e. the torsion) of Ω. Denote by θ a primitive of τ and suppose that $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$. Assume that Assumptions 3.1 to 3.3 are satisfied. Let H_k be the operator defined by (3.2)-(3.7). Let $f \in L^2(\Omega_0)$ be a non null function and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of $H_k\phi = f$ in Ω_0. Assume that there exists a positive constant M such that $|f(s,u)| \leq M|\phi(s,u)|$ almost everywhere in Ω_0.*
Then the data (f, ϕ) determines uniquely the first curvature function k if the torsion τ is given.

4 Proofs of Theorem 3.1, 3.2 and 3.3

4.1 Proof of Theorem 3.1

Recall that ϕ is an eigenfunction of H_k. Since ϕ is continuous, $H_k\phi = \lambda\phi$ and $\phi \in H^2(\Omega_0)$ then $H_k\phi$ is continous. Therefore, for $u = (u_2, u_3) = (0,0)$, we get: $-\Delta\phi(s,0,0) - \frac{k^2(s)}{4}\phi(s,0,0) = \lambda\phi(s,0,0)$ and equivalently, $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\lambda$ if $\phi(s,0,0) \neq 0$.

4.2 Proof of Theorem 3.2

We follow the proof of Theorem 1.2. We have $H_k\phi = f$ with $\phi \in H_0^1(\Omega_0)$. So

$$\int_{\Omega_0} [h^{-2}(\partial_s\phi)(\partial_s\psi) + (\partial_{u_2}\phi)(\partial_{u_2}\psi) + (\partial_{u_3}\phi)(\partial_{u_3}\psi)] = \int_{\Omega_0} [f - V_k\phi]\psi \text{ for all } \psi \in H_0^1(\Omega_0) \quad (4.1)$$

with h defined by (3.2) and V_k defined by (3.7).

From Assumptions 3.2 and 3.3, since $k, k', k'', \theta', \theta''$ are bounded, we deduce that $V_k \in L^\infty(\Omega_0)$. Therefore $f - V_k\phi \in L^2(\Omega_0)$. Moreover we have also $h^{-2} \in C^1(\overline{\Omega_0})$ and $D^\alpha(h^{-2}) \in L^\infty(\Omega_0)$ for any α, $|\alpha| \leq 1$. Thus, using Lemma 2.1 for the equation (4.1), we obtain that $\phi \in H^2(\Omega_0)$.
By the same way, we get that $f - V_k\phi \in H^1(\Omega_0)$, $h^{-2} \in C^2(\overline{\Omega_0})$ and $D^\alpha(h^{-2}) \in L^\infty(\Omega_0)$ for any α, $|\alpha| \leq 2$ (since $k \in C^3(\mathbb{R})$, $\theta \in C^3(\mathbb{R})$ and all of their derivatives are bounded). Using Lemma 2.1, we obtain that $\phi \in H^3(\Omega_0)$.
We apply again the Lemma 2.1 to get that $\phi \in H^4(\Omega_0)$ (since $f - V_\gamma\phi \in H^2(\Omega_0)$, $c_\gamma \in C^3(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for all α, $|\alpha| \leq 3$, from the hypotheses $\gamma \in C^4(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for $k = 0, \ldots, 4$.).
Finally, using Assumption 3.4 and Lemma 2.1, we obtain that $\phi \in H^5(\Omega_0)$. Due to the regularity of Ω_0 (see [1, Note p.169]), we have $\phi \in H^5(\mathbb{R}^3)$ and $\Delta\phi \in H^3(\mathbb{R}^3)$. Since $\nabla(\Delta\phi) \in (H^2(\mathbb{R}^3))^3$ and $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, we can deduce that $\Delta\phi$ is continuous (see [1, Remark 8 p.154]).
Thus we conclude as in Theorem 1.2 and for $u = (u_2, u_3) = (0, 0)$, we get: $-\Delta\phi(s, 0, 0) - \frac{k^2(s)}{4}\phi(s, 0, 0) = f(s, 0, 0)$ and equivalently, $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\frac{f(s,0,0)}{\phi(s,0,0)}$ if $\phi(s, 0, 0) \neq 0$.

4.3 Proof of Theorem 3.3

We prove here that (f, ϕ, θ) determines uniquely k.
Assume that Ω_1 and Ω_2 are two guides in $\mathbb{R}^3$. We denote by k_1 and k_2 the first curvatures functions associated with Ω_1 and Ω_2 and we denote by θ a primitive of τ the common torsion of Ω_1 and Ω_2. We suppose that k_1, k_2 and θ satisfy the Assumptions 3.2 and 3.3 and that $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$. Assume that $H_{k_1}\phi = f = H_{k_2}\phi$.
Then ϕ satisfies

$$\begin{aligned} &-\partial_s((h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3))\partial_s\phi(s, u_2, u_3)) \\ &+(V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3))\phi(s, u_2, u_3) = 0 \end{aligned} \tag{4.2}$$

where h_1 (associated with k_1) is defined by (3.2), V_{k_1} is defined by (3.7), h_2 (associated with k_2) is defined by (3.2) and V_{k_2} is defined by (3.7).
Assume that $k_1 \not\equiv k_2$.

Step 1. First, we consider the case where (for example) $k_1(s) < k_2(s)$ for all $s \in \mathbb{R}$. Recall that each k_i is a nonnegative function.
Let $\epsilon > 0$ and denote by $J_\epsilon :=]-\epsilon, 0[\times]-\epsilon, 0[$, $O_\epsilon := \mathbb{R} \times J_\epsilon$ with ϵ small enough to have $J_\epsilon \subset \omega$ (recall that $\Omega_0 = \mathbb{R} \times \omega$).
Multiplying (4.2) by ϕ and integrating over O_ϵ, we get:

$$\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s\phi)^2 + \int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s\phi)\phi\nu_s + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 = 0. \tag{4.3}$$

Since $\epsilon << 1$, $V_{k_i} \simeq -\frac{k_i^2(s)}{4}$ for $i = 1, 2$, and so $V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3) > 0$ in O_ϵ.
Moreover, note that:

$$h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3) = \frac{\alpha(s, u_2, u_3)(k_1(s) - k_2(s))(h_1(s, u_2, u_3) + h_2(s, u_2, u_3))}{h_1^2(s, u_2, u_3)h_2^2(s, u_2, u_3)} \tag{4.4}$$

with $\alpha(s, u_2, u_3) := \cos(\theta(s))u_2 + \sin(\theta(s))u_3$.
Since $(u_2, u_3) \in J_\epsilon$ and $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$, we have $\alpha(s, u_2, u_3) < 0$. Therefore, by (4.4), we deduce that $h_1^{-2} - h_2^{-2} > 0$ in O_ϵ.

Thus $\int_{O_\epsilon}(h_1^{-2}-h_2^{-2})(\partial_s\phi)^2+\int_{O_\epsilon}(V_{k_1}-V_{k_2})\phi^2\geq 0$.
Note also that:

$$\int_{\partial O_\epsilon}(h_1^{-2}-h_2^{-2})(\partial_s\phi)\phi\nu_s=0. \tag{4.5}$$

Therefore, from (4.3) and (4.5) we get:

$$\int_{O_\epsilon}(h_1^{-2}-h_2^{-2})(\partial_s\phi)^2+\int_{O_\epsilon}(V_{k_1}-V_{k_2})\phi^2=0 \tag{4.6}$$

with $h_1^{-2}-h_2^{-2}>0$ in O_ϵ and $V_{k_1}-V_{k_2}>0$ in O_ϵ.
From (4.6) we can deduce that $\phi=0$ in O_ϵ. Using a unique continuation theorem (see [10, Theorem XIII.63 p.240]), from $H_{k_1}\phi=f$, noting that $-\Delta(W_g^{-1}\phi)=W_g^{-1}f=g^{-1/4}f$, by $|f|\leq M|\phi|$ a.e. in Ω_0, we can deduce that $\phi=0$ in Ω_0. So we get a contradiction since f is assumed to be a non null function.

Step2. From Step 1, we obtain that there exists at least one point $s_0\in\mathbb{R}$ such that $k_1(s_0)=k_2(s_0)$. Since $k_1\not\equiv k_2$, we can choose $a\in\mathbb{R}$ and $b\in\mathbb{R}\cup\{+\infty\}$ such that (for example) $k_1(a)=k_2(a)$, $k_1(s)<k_2(s)$ for all $s\in]a,b[$ and $k_1(b)=k_2(b)$ if $b\in\mathbb{R}$. We proceed as in Step 1, considering in this case $O_\epsilon:=]a,b[\times J_\epsilon$. From $k_1(a)=k_2(a)$, we get that $h_1^{-2}(a,u_2,u_3)=h_2^{-2}(a,u_2,u_3)$. Therefore we obtain $\int_{\partial O_\epsilon}(h_1^{-2}-h_2^{-2})(\partial_s\phi)\phi\nu_s=0$. So (4.3) becomes (4.6) with $h_1^{-2}-h_2^{-2}>0$ in O_ϵ and $V_{k_1}-V_{k_2}>0$ in O_ϵ. So $\phi=0$ in O_ϵ and as in Step 1, by a unique continuation theorem, we obtain that $\phi=0$ in Ω_0. Therefore we get a contradiction.

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